Multinomial Theorem

Multinomial Theorem is a natural extension of binomial theorem and the proof gives a good exercise for using the Principle of Mathematical Induction.

Theorem Let P(n) be the proposition:

$$(x_1 + x_2 + \ldots + x_n)^N = \sum_{\substack{i_1, i_2, \ldots, i_n \geq 0 \\ i_1 + i_2 + \ldots + i_n = N}} \left[\frac{N!}{i_1! i_2! \ldots i_n!} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \right] \quad \text{where} \quad n, N \in \mathbf{N}.$$

Proof P(1) is obviously true. For P(2), By Binomial Theorem

$$(x_1 + x_2)^N = \sum_{r=0}^{N} C_r^n x_1^{N-r} x_2^{r} = \sum_{r=0}^{N} \frac{N!}{(N-r)!\, r!} x_1^{N-r} x_2^{r} \text{, by Binomial Theorem}$$

$$= \sum_{i_2=0}^{N} \frac{N!}{i_1!\, i_2!} x_1^{i_1} x_2^{i_2}, \quad \text{where} \quad i_1 = N - r, \quad i_2 = r$$

$$= \sum_{\substack{i_1, i_2 \geq 0 \\ i_1 + i_2 = N}} \left[\frac{N!}{i_1! i_2} x_1^{i_1} x_2^{i_2} \right] \qquad \therefore \quad P(2) \text{ is true.}$$

Assume P(k) is true for some $k \in \mathbf{N}$, that is

$$(x_1 + x_2 + \ldots + x_k)^N = \sum_{\substack{i_1, i_2, \ldots, i_k \geq 0 \\ i_1 + i_2 + \ldots + i_k = N}} \left[\frac{N!}{i_1! i_2! \ldots i_k!} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} \right] \qquad \ldots. \quad (1)$$

For P(k + 1),

$$(x_1 + x_2 + \ldots + x_k + x_{k+1})^N = [(x_1 + x_2 + \ldots + x_k) + x_{k+1}]^N$$

$$= \sum_{r=0}^{N} C_r^n (x_1 + x_2 + \ldots + x_k)^{N-r} x_{k+1}^{r} = \sum_{r=0}^{N} C_r^n \sum_{\substack{i_1, i_2, \ldots, i_k \geq 0 \\ i_1 + i_2 + \ldots + i_k = N-r}} \left[\frac{(N-r)!}{i_1! i_2! \ldots i_k!} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} \right] x_{k+1}^{r}$$

$$= \sum_{r=0}^{N} \sum_{\substack{i_1, i_2, \ldots, i_k \geq 0 \\ i_1 + i_2 + \ldots + i_k = N-r}} \left[\frac{N!}{r!\,(N-r)!} \frac{(N-r)!}{i_1! i_2! \ldots i_k!} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} x_{k+1}^{r} \right]$$

$$= \sum_{i_{k+1}=0}^{N} \sum_{\substack{i_1, i_2, \ldots, i_k \geq 0 \\ i_1 + i_2 + \ldots + i_k = N - i_{k+1}}} \left[\frac{N!}{i_1! i_2! \ldots i_k!\, i_{k+1}!} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} x_{k+1}^{i_{k+1}} \right] \text{, where} \quad r = i_{k+1}.$$

$$= \sum_{\substack{i_1, i_2, \ldots, i_k, i_{k+1} \geq 0 \\ i_1 + i_2 + \ldots + i_k + i_{k+1} = N}} \left[\frac{N!}{i_1! i_2! \ldots i_k! i_{k+1}!} x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k} x_{k+1}^{i_{k+1}} \right] \qquad \therefore \quad P(k+1) \text{ is true.}$$

By the Principle of Mathematical Induction, P(n) is true $\forall\, n \in \mathbf{N}$.